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Two-way traffic flow: Exactly solvable model of traffic jam

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Abstract. We study completely asymmetric two-channel exclusion processes in one dimension. It describes a two-way traffic flow with cars moving in opposite directions. The interchannel interaction makes cars slow down in the vicinity of approaching cars in the other lane. Particularly, we consider in detail the system with a finite density of cars on one lane and a single car on the other. When the interchannel interaction reaches a critical value, a traffic jam occurs, which turns out to be of first-order phase transition. We derive exact expressions for the average velocities, the current, the density profile and the k-point density correlation functions. We also obtain the exact probability of two cars being in one lane of distance R apart, provided there is a finite density of cars on the other lane, and show that the two cars form a weakly bound state in the jammed phase.

1. Introduction

Low-dimensional systems out of equilibrium have attracted much attention recently [1]. An important class of such models is the one-dimensional (1D) exclusion processes describing particles hopping independently with hard-core repulsion along a 1D lattice. Such systems provide a good description of growth processes, traffic flow and queueing problems [2], etc (see [1] for the references up to 1995). The completely asymmetric exclusion process (ASEP) describing particles hopping only to the right with equal rate 1 and hard-core repulsion is perhaps the simplest and best studied one [3, 4]. In particular, for the periodic boundary condition, all configurations are equally likely in the steady state, and the average particle velocity in the infinite system is

$$\langle v\rangle = 1 - n \tag{1}$$

n being the density of particles. Janowsky and Lebowitz [5] showed that a fixed blockage in the system, which reduces the rate of hopping across it from 1 to $r < 1$, can produce global effects. Namely, for each fixed density n, there is a range of $0 < r \leqslant r_0$ where the system segregates in high- and low-density regions with a sharp boundary, called shock, between them. Although some exact results were obtained [5], many quantities of interest, for example steady-state density profile, correlation functions, etc were computed only numerically.

In this paper we derive all these quantities exactly, in closed form, for a slightly different model, guided by modelling the two-way traffic-flow problem. Namely, there are two 1D chains on a ring, N sites each. One chain, or lane is occupied by cars and another is

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occupied with trucks (we refer to them differently just for notational simplicity), hopping in opposite directions with rates 1 and γ, respectively. Effective rate of hopping of the car (truck) reduces to $1/\beta(\gamma/\beta)$, when there is a truck (car) in front in another lane. Physically, $1/\beta$ is determined by the narrowness of the road; it describes how much the car/truck slows down when seeing another truck/car approaching. The $\frac{1}{\beta} = 0$ case corresponds to the road being completely blocked.

For the case where there is a single truck in one lane and finite density of cars n in the other, we expect the similar type of behaviour with the blockage case [5]. We expect the cars to pile up causing a traffic jam, at a certain range of interlane interaction parameter β. So it is, as Monte Carlo (MC) simulations unambigiously show, see figure 2. Then, to study closely the nature of traffic jam phase transition, we impose the restriction that the car and truck cannot occupy each neighbouring site i simultaneously. At this point, the model becomes exactly solvable by the matrix approach of Derrida *et al* [6, 7]. Using it, we compute the average velocities, the density profile, and k-point correlation functions exactly, for the finite chain and in the thermodynamic limit. Particularly, the traffic jam phase transition curve is given by simple formula $\beta_{\mathrm{crit}} = 1/n$, in the thermodynamic limit.

Characterictics of the traffic jam phase transition are examined in detail. Finally, we consider the situation with two trucks/finite density of cars in the system. We observe that a weak bound state is formed between the two trucks.

A comparison of our results with the original two-way model, the ASEP with single fixed blockage [5], and exact Bethe ansatz solution by Schütz [8] for deterministic ASEP with blockage concludes the paper.

2. Formulation of the two-way traffic-flow model

We consider the following hard-core exclusion process: there are two parallel 1D chains on a ring, N sites each, the first chain contains M cars and the second chain contains K trucks. Cars (trucks) are hopping in opposite directions with rates $1(\gamma)$ respectively. The state of the system is characterized by the set of occupation numbers $\{\tau_i\}_{i=1}^{N}$ of the first lane and $\{\sigma_i\}_{i=1}^{N}$ of the second lane. $\tau_i = 1$ if there is a car at site i and $\tau_i = 0$ if empty, and the same for the trucks, $\sigma_i = 1(0)$ if occupied (empty). The system evolves under the following stochastic dynamical rules.

At each infinitesimal time interval dt, one pair of adjacent sites $i, i+1$ at any of the two chains is selected at random for a possible exchange of states. The possible exchange processes together with their rates are listed below: cars are hopping to the right

$$(\tau_i, \tau_{i+1}) = (1, 0) \to (0, 1) \qquad \text{with rate } \begin{cases} 1 & \text{if } \sigma_{i+1} = 0 \\ \dfrac{1}{\beta} & \text{if } \sigma_{i+1} = 1 \text{ (truck in front)} \end{cases} \tag{2}$$

trucks are hopping to the left

$$(\sigma_i, \sigma_{i+1}) = (0, 1) \to (1, 0) \qquad \text{with rate } \begin{cases} \gamma & \text{if } \tau_i = 0 \\ \dfrac{\gamma}{\beta} & \text{if } \tau_i = 1 \text{ (car in front).} \end{cases} \tag{3}$$

One sequence of $2N$ (total number of sites in two chains) selections constitutes one time step (or one MC step, see figures 1 and 2).

The interlane interaction parameter $\beta > 1$ has a transparent physical meaning. It describes how much a vehicle slows down when seeing another vehicle approaching, which in turn depends on the narrowness of the road. $\beta = 1$ (no slowing down) corresponds

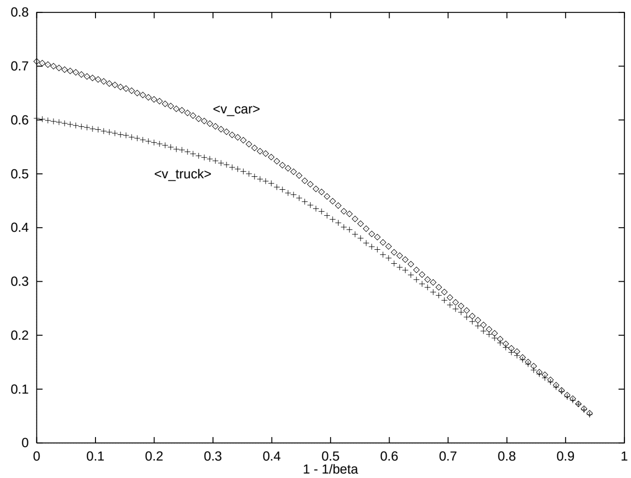

Figure 1. MC simulation result of the two-lane traffic flow model. The 'average velocities versus interlane interaction $r = 1 - \frac{1}{\beta}$' are shown for $N = 200$, $n_{\text{cars}} = 0.3$, $n_{\text{trucks}} = 0.4$ and $\gamma = 1$, unless otherwise stated. Initial configuration of the system is random. We equilibriate the system for 2000 MC step intervals, collect data at 2000 MC step intervals and average over 100 different histories.

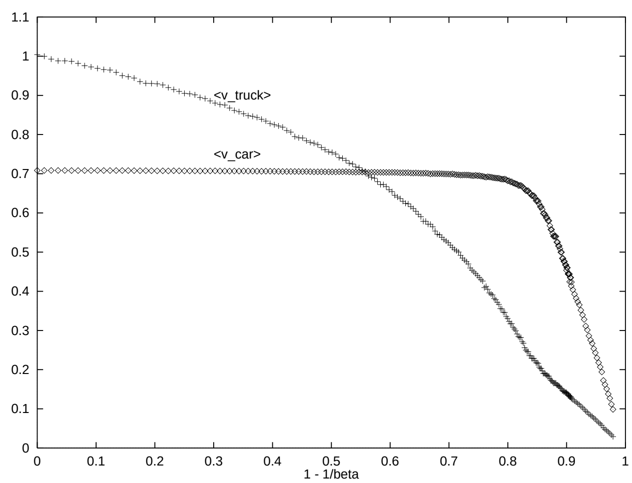

Figure 2. The same as in figure 1 but for $N_{\text{cars}} = M = 60$ and $N_{\text{trucks}} = 1$. The point $r_c \approx 0.8$ is the approximate traffic jam transition point.

to a highway with a divider, and $1/\beta = 0$ corresponds to a narrow road being completely blocked. If we let the system evolve for a long time, it reaches the steady state, independent of its initial configuration. We shall be interested in the steady-state characteristics which depend only on macroscopic parameters (number of cars, number of trucks and total number of sites) and the rates (2), (3).

We studied the most practical characteristics of the model, the average velocities of cars $\langle v_{\text{car}} \rangle$ and trucks $\langle v_{\text{truck}} \rangle$ as a function of $(1 - 1/\beta)$ by MC simulations. The MC results for two different cases are shown in figures 1 and 2. Figure 1 corresponds to a system where the density of cars and trucks are 0.3 and 0.4, respectively, and shows a monotonic decrease of both velocities as β increases. Figure 2 corresponds to the system with a single truck and many cars with density $n = 0.3$. In contrast to figure 1, we see that $\langle v_{\rm car}\rangle$ remains constant (equal to the average velocity (1) in the noninteracting system $\langle v\rangle = (1-n)$), until the point $1 - \frac{1}{\beta_c} \approx 0.8$ is reached. For $\beta > \beta_c$, $\langle v_{\rm car}\rangle$ rapidly drops. Simulations show that for $\beta > \beta_c$, the system segregates into two phases: the high-density one in front of the truck (traffic jam) and the low-density one behind the truck. Piling up of cars in front of the truck accounts for the decrease of average car velocity. The absence of a sharp transition for a finite density of trucks (figure 1) compared with figure 2 is due to the fact that a finite number of trucks in infinite systems produces the macroscopic jammed phase, while a finite density of trucks produces only microscopic jams which average out to give a smooth behaviour.

The segregated or traffic jam phase is well known as a shock phase or coexistence phase in 1D ASEP. Schütz [8] showed its existence in an exactly solvable deterministic model with a fixed blockage, and found various shock characteristics rigorously. As far as the probabilistic ASEP are concerned, Janowsky and Lebowitz [5] showed the existence of the segregated phase in a probabilistic ASEP with a fixed blockage. The latter model is not solvable, and most results obtained in [5] are therefore numerical.

The characteristics of the shock are believed to be quite universal, qualitatively independent of details of the stochastic process. That is why it is important to give exact solutions for some system with a shock. Here in this paper we propose a two-way traffic-flow problem (slightly modified, see below) with a single truck as an example of such a solvable system. Roughly, the single truck plays the role of blockage and $\frac{1}{\beta}$ plays the role of the transmission coefficient r in [5]. Following the traffic-flow formulation, we shall call the shock the 'traffic jam' and the segregated or coexistence phase the 'traffic jam phase'. We shall find exactly the characteristics of the traffic jam and the traffic jam transition, including average velocities, density profiles, k-point correlation functions, for finite chains and in the thermodynamic limit. For this purpose, one has to modify the original model to a solvable one.

3. Modification of the original model to an exactly solvable model

We now modify the two-way traffic problem slightly. Here, we forbid a car and a truck to occupy two parallel sites i in the neighbouring chains simultaneously. Then one can actually describe the configuration by a single-lane configuration $\{\tau_i\}_{i=1}^N$; each site i is either occupied by a car $\tau_i = 1$ or truck $\tau_i = 2$ or empty $\tau_i = 0$. The allowed exchange processes are then modified from equations (2) and (3) to:

$$\begin{aligned}
&(1,0) \to (0,1) \qquad &&\text{with rate } 1\\
&(0,2) \to (2,0) \qquad &&\text{with rate } \gamma\\
&(1,2) \to (2,1) \qquad &&\text{with rate } \tfrac{1}{\beta}.
\end{aligned} \tag{4}$$

Although the quantitative characteristics of the system do change after this modification, the qualitative characteristics do not (compare for instance the graphs for average car velocities in figures 1 and 3).

Process (4) is the two-species ASEP solvable by the approach of Derrida *et al* [7]. Process (4) and the one considered in [7] differ by replacement $2 \leftrightarrow 0$ (interchange of trucks and empty spaces).

The probability of a given steady-state configuration is shown in [7] to be proportional

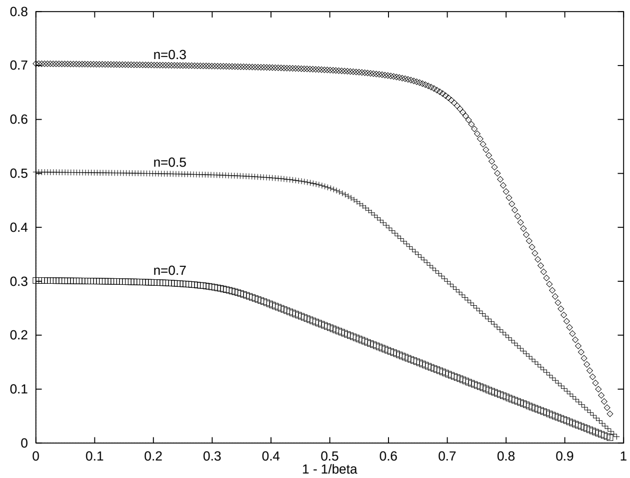

Figure 3. Average velocities of the cars computed from equation (8) in a system of 200 sites, for different densities $n = 0.3$, 0.5, 0.7.

to the trace of a product

$$w_{\text{conf}}(\tau_1\tau_2\ldots\tau_N) = \mathrm{Tr}(X_1X_2\ldots X_N) \tag{5}$$

where

$$X_i = \begin{cases} D & \text{if car at site } i \qquad \tau_i = 1 \\ E & \text{if truck at site } i \qquad \tau_i = 2 \\ A & \text{if site } i \text{ is empty} \qquad \tau_i = 0 \end{cases} \tag{6}$$

are noncommuting matrices satisfying the following algebra:

$$\begin{aligned} &DE = D + E \\ &\beta DA = A \\ &\alpha AE = A; \qquad \alpha = \beta\gamma. \end{aligned} \tag{7}$$

Knowing the probabilistic measure, we can find the various averages of steady state.

4. Average velocities

We shall consider the system having M number of cars and a single truck.

Analogously to [7], define $Y(N, M)$ as the probability of having the truck at site N, in a system with N sites and M cars. Define then $Y_D(N, M)$ as the probability of finding a car at site $N - 1$, provided the truck occupies the position N. Then, the average velocities of the cars and the truck are given by:

$$\langle v_{\text{car}}\rangle = \frac{1}{\beta}\frac{Y_D(N, M) + (N - M - 1)Y(N - 1, M - 1)}{MY(N, M)} \tag{8}$$

$$\langle v_{\text{truck}}\rangle = \frac{1}{\beta}\frac{Y_D(N, M) + \alpha(Y(N, M) - Y_D(N, M))}{Y(N, M)}. \tag{9}$$

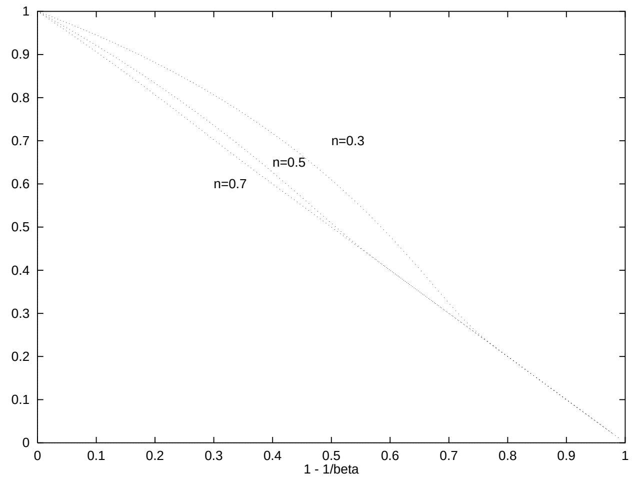

Figure 4. The same as in figure 3 for the the truck velocities equation (9).

The quantities $Y(N, M)$ and $Y_D(N, M)$ are computed in the appendix and found to be†

$$Y_D(N, M) = \frac{1}{\alpha\beta^M}\left(\frac{-\alpha}{\beta-1}C_{N-2}^{M-1} + \frac{\alpha+\beta-1}{\beta-1}I(N, M)\right) \tag{10}$$

$$Y(N, M) = Y_D(N, M) + \frac{1}{\alpha\beta^M}C_{N-2}^{M} \tag{11}$$

where

$$I(N, M) = \sum_{k=1}^{M}\beta^k C_{N-2-k}^{M-k} \tag{12}$$

and C_i^j is the binomial coefficient. In figures 3 and 4 we plot $\langle v_{\rm car}\rangle$ and $\langle v_{\rm truck}\rangle$, respectively, computed from an exact formula, as a function of $1 - 1/\beta$, for three densities $n = 0.3$, 0.5 and 0.7 at $\gamma = 1$, in a system with $N = 200$ sites. Naturally the traffic jam transition point decreases as the average density n increases. The behaviour of $\langle v_{\rm car}\rangle$ is similar to the one for the original two-way traffic model in figure 2. However, now we can evaluate the exact thermodynamic limit $N, M \to \infty$, $n = M/N$ fixed, and find the exact transition point. We used the steepest descent method for computing the thermodynamic limits. The average velocities in the thermodynamic limit are given by (see figure 5):

$$\langle v_{\rm car}\rangle = \begin{cases} 1-n & \text{if } n\beta \leqslant 1 \\ \dfrac{1}{\beta}\dfrac{1-n}{n} & \text{if } n\beta \geqslant 1 \end{cases} \tag{13}$$

$$\langle v_{\rm truck}\rangle = \begin{cases} \dfrac{1}{\beta}\;\dfrac{\alpha(1-n)(1-n\beta)+n(\alpha+\beta-n\beta)}{(1-n)(1-n\beta)+n(\alpha+\beta-n\beta)} & \text{if } n\beta \leqslant 1 \\ \dfrac{1}{\beta} & \text{if } n\beta \geqslant 1. \end{cases} \tag{14}$$

Thus, the transition point to the jammed state is given by a simple formula

$$n\beta_{\rm crit} = 1. \tag{15}$$

† More precisely, $Y(N, M)$ and $Y_D(N, M)$ are probabilities up to a normalization factor which is equal for all the terms entering equations (8) and (9), see the appendix.

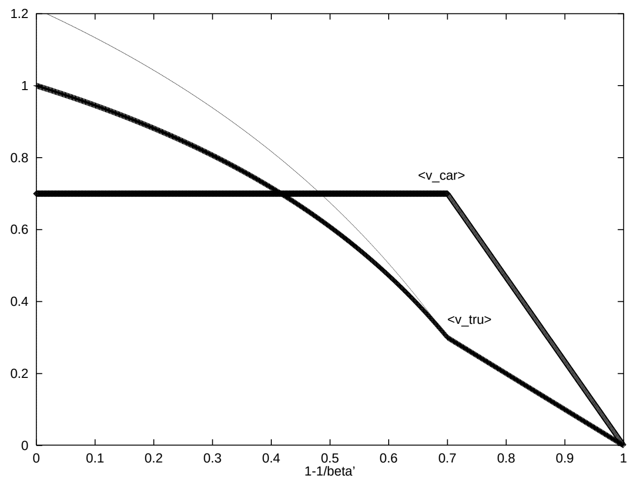

Figure 5. Exact car and truck velocities in the thermodynamic limit equations (13) and (14), for $n = 0.3$. $\langle v_{\text{truck}} \rangle$ is given for $\gamma = 1$ and $\gamma = 1.5$ (light curve).

Note that the transition point does not depend on γ—the free velocity of the truck. The average car velocity has a cusp at the transition point. The average velocity of cars before the transition $n\beta < 1$ is equal to the one in a system without a truck (1). For $n = 1$, $\langle v_{\text{car}} \rangle \equiv 0$ independently of β, because all sites are filled and cars cannot move. $1/\beta = 0$ is the case of complete blockage: both velocities are identically zero.

To examine closely the nature of the traffic jam transition, we find the exact density profile and k-point correlation functions in the next two sections.

5. The density profile

In this section, we obtain the exact density profile $\langle n(x) \rangle$ in a system with one truck and arbitrary M number of cars, in a chain of length N. We choose a reference frame in which the truck is always at the position N. It can be done because the weights of the steady-state configurations (5) depend only on the positions of the cars relative to the truck location, due to cyclic invariance of (5). The average density $\langle n(x) \rangle$ at distance x from the truck is equal to the probability of finding a car at site $N - 1 - x$;

$$\langle n(x) \rangle = \frac{\sum_{\text{conf}} \tau_{N-1-x} w_M(\tau_1 \tau_2 \ldots \tau_{N-1} 2)}{\sum_{\text{conf}} w_M(\tau_1 \tau_2 \ldots \tau_{N-1} 2)}. \tag{16}$$

Sums run over all possible configurations having M cars, $N - M - 1$ empty spaces, with a truck at the position N, so

$$w_M = \operatorname{Tr}\left(\left(\prod_{i=1}^{N-1} X_i\right) E\right) \tag{17}$$

where $X_i = D(A)$ if site i is occupied by a car (empty). The quantity (16) is readily obtained using algebra (7) and found to be

$$\langle n(x) \rangle = \left\{ C_{N-3}^{M-1} - \frac{\alpha}{\beta - 1} C_{N-3}^{M-2} + \frac{\alpha + \beta - 1}{\beta - 1} [I(N-1, M-1) \right.$$
$$\left. + \beta^{x-1} (\beta - 1) I(N - x, M - x) \Theta(M \geqslant x)] \right\} / (Y(N, M) \alpha \beta^M) \tag{18}$$

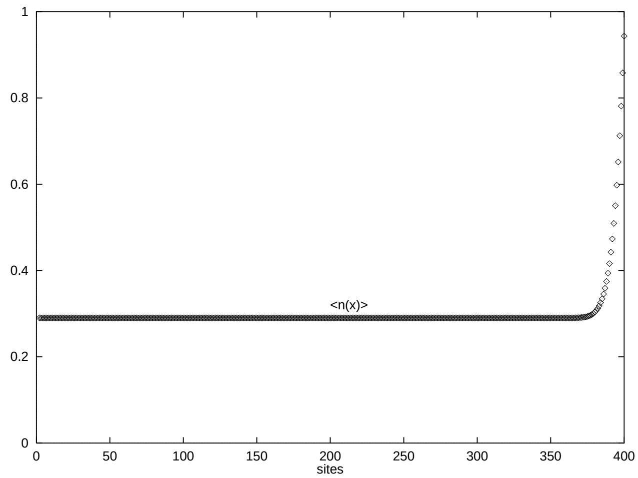

Figure 6. Density profile before traffic jam, in a chain of 400 sites computed from formula (18). $n = \frac{M}{N} = 0.3$, $\beta = 3(\beta_{\text{crit}} = 3.3333)$. The single truck is located at the right end.

where $Y(N, M)$ and $I(N, M)$ are given by equations (11) and (12), respectively, and

$$\Theta(y \geqslant x) = \begin{cases} 1 & \text{if } y \geqslant x \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Expression (18) determines the average density for any value of β. Below we shall consider the cases before and after the traffic jam phase transition separately.

5.1. Low-density phase; $n\beta < 1$

Before the transition, the presence of a truck affects the system only locally as is seen from figure 6, where the density profile is shown for $N = 400$. The density, otherwise constant, locally increases only in a close vicinity of the truck.

Evaluating formula (18) for large $M, N \gg 1$; with $\frac{M}{N} = n$ fixed, we find

$$\langle n(x) \rangle = n\left(1 + \frac{(\alpha + \beta - 1)(1 - n)}{1 - n + \alpha n}(n\beta)^x\right). \tag{20}$$

One sees that the local density disturbance decays exponentially at a finite-length scale

$$\xi = |\ln(n\beta)|^{-1}. \tag{21}$$

Therefore, the relative size of the disturbed region vanishes as $\frac{1}{N}$. In principle, by common sense one would expect the existence of the low-density region right behind the truck. However, it is absent in the exact solution, as seen from figure 6. We do not have a simple explanation for this fact. Analogous behaviour was observed in an exactly solvable deterministic exclusion process with a fixed blockage [8].

5.2. Traffic jam phase; $n\beta > 1$

In this region, cars pile up before the truck as seen from figure 7. Increasing interlane interaction β leads to increase of traffic jam length

$$l = \frac{L_{\text{jam}}}{N} \approx \frac{n\beta - 1}{\beta - 1}. \tag{22}$$

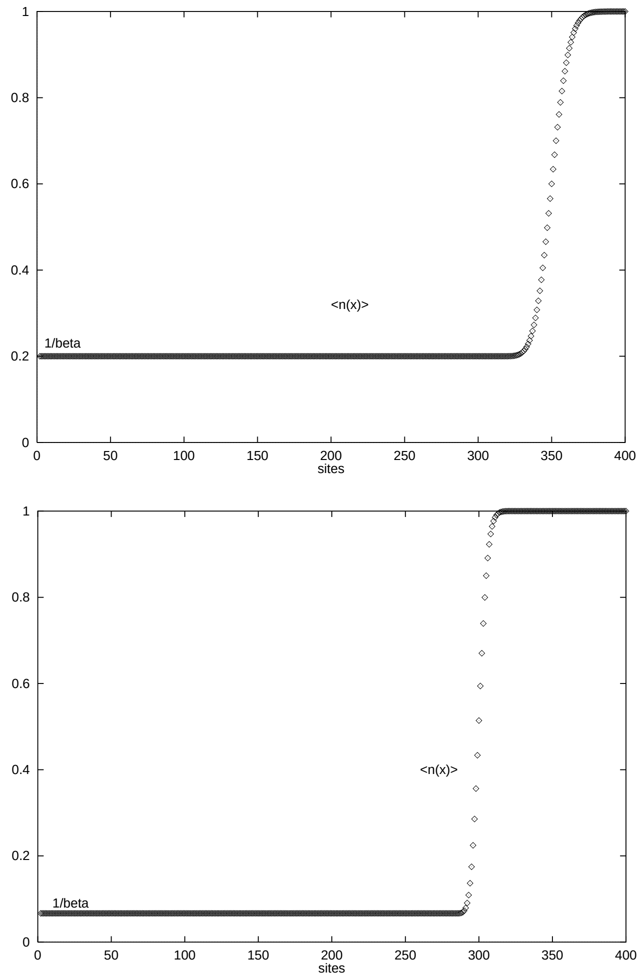

Figure 7. Density profile in the traffic jam phase, in a chain of 400 sites computed from formula (18) for $n = \frac{M}{N} = 0.3$, and for $\beta = 5$(a) and $\beta = 15(b)$. ($\beta_{\rm crit} = 3.3333$). The total length of traffic jam is $L_{\rm jam} \approx \frac{n\beta-1}{\beta-1} N$. Asymptotic values of densities in a high- (low-)density region are $n_{\rm high} = 1$, $n_{\rm low} = \frac{1}{\beta}$. The truck is located at the right end.

In the thermodynamic limit $M, N \to \infty$, $\frac{M}{N} = n$, the latter formula becomes exact, and the density profile becomes a step function:

$$\langle n(x) \rangle = \begin{cases} 1 & \frac{x}{N} \leqslant \frac{n\beta-1}{\beta-1} \\ \dfrac{1}{\beta} & \text{otherwise.} \end{cases} \tag{23}$$

Note that the density in the low-density region is equal to the critical density $n_{\rm crit} = 1/\beta$, independently of the average density n. The same behaviour is observed in [8].

In fact, in the jammed phase, the only way the truck can move is by the process $(1, 2) \to (2, 1)$ (because the contribution of the processes $(0, 2) \to (2, 0)$ becomes exponentially small in the large N limit). In the slow truck/many cars problem treated in [9], the same is true when the rate for the $(2, 0) \to (0, 2)$ process as denoted by α in [9] is zero. Thus our model in the jammed case is a special case of [9], up to exponentially small corrections. However, only simple characteristics were investigated in [9]; correlation functions, as well as large $N \gg 1$ limits were not studied.

More precisely, for large $N, M \gg 1$, we find up to corrections of order $N^{-\frac{1}{2}}$,

$$\langle n(x)\rangle = 1 - \frac{1}{2}\left(1 - \frac{1}{\beta}\right)\left\{1 + \operatorname{erf}\left(\frac{x - Nl}{\Delta\sqrt{N}}\right)\right\} + \mathrm{O}(N^{-\frac{1}{2}}) \tag{24}$$

with $l = \frac{n\beta-1}{\beta-1}$, $\Delta = \frac{\sqrt{2\beta(1-n)}}{\beta-1}$ and $\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y \mathrm{e}^{-t^2}\mathrm{d}t$.

This shows that the shock interface extends over a region of width $\sqrt{N}$. More careful considerations, however, show that the real shock interface is sharp and extends over only two consecutive sites, and the apparent width of $\sqrt{N}$ is due to shock-position fluctuations. (The shock-position fluctuations of order $\sqrt{N}$ were also observed in the probabilistic exclusion process with a fixed blockage [5].) Indeed, the discrete version of the density gradient correlation, $\langle \Delta n(x_1)\Delta n(x_2)\rangle$ where $\Delta n(x) = n(x+1) - n(x)$ vanishes if $|x_1 - x_2| > 1$. (It follows directly from equations (31) and (32).) This shows that the jammed phase is indeed segregated into two macroscopic regions—the low-density one on the left with $n_{\mathrm{low}} = \frac{1}{\beta}$ and the high-density one on the right $n_{\mathrm{high}} = 1$. The fact that $n_{\mathrm{high}} = 1$ is due to the nature of the process we consider (see (4)): once cars pile up before the truck, the car–truck exchange processes do not create empty spaces.

Finally, note that one can choose the difference between the average densities in the macroscopic regions in front and behind the truck $\delta n = n_{\mathrm{high}} - n_{\mathrm{low}}$ as an order parameter, characterizing the traffic jam phase transition. With respect to this order parameter, the transition to the jammed phase is of the first order, as seen from equation (23);

$$\delta n = \begin{cases} 0 & \text{if } n\beta < 1 \\ 1 - \dfrac{1}{\beta} & \text{if } n\beta \geqslant 1. \end{cases} \tag{25}$$

5.3. *The hydrodynamic approach*

The thermodynamic limit results equation (23) can also be obtained from simple hydrodynamic arguments. Supposing that the segregated phase contains two macroscopic regions of length lN and $(1-l)N$, with average car velocities in these regions $1 - n_{\mathrm{high}}$ and $1 - n_{\mathrm{low}}$, respectively (see equation (1)), one can write down a set of equations. First, from the car conservation,

$$n_{\mathrm{low}}(1-l) + n_{\mathrm{high}}l = n$$

and next, from the current conservation in the reference frame of the fixed truck,

$$j = n_{\mathrm{low}}(1 - n_{\mathrm{low}} + \langle v_{\mathrm{truck}}\rangle) = n_{\mathrm{high}}(1 - n_{\mathrm{high}} + \langle v_{\mathrm{truck}}\rangle) \tag{26}$$

finally from the definition of the average car velocity

$$n\langle v_{\mathrm{car}}\rangle = n_{\mathrm{low}}(1-l)(1-n_{\mathrm{low}}) + n_{\mathrm{high}}l(1-n_{\mathrm{high}}).$$

Substituting the values $\langle v_{\mathrm{car}}\rangle$ and $\langle v_{\mathrm{truck}}\rangle$ from equations (13) and (14), and solving the above system of three equations, we obtain exactly the result equation (23).

The correctness of the hydrodynamic arguments in the thermodynamic limit is due to the fact that indeed cars behave like an ideal gas of interacting particles; correlations vanish in the thermodynamic limit as shown in the next section.

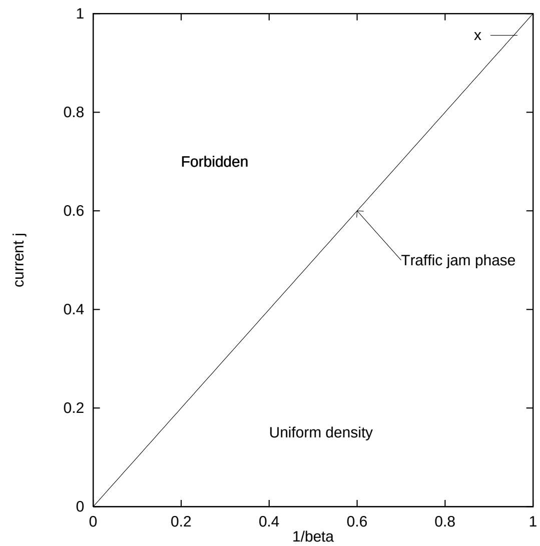

Figure 8. Phase diagram of the solvable model in 'current versus $\frac{1}{\beta}$' plane. The curve $j = \frac{1}{\beta}$ corresponds to jammed phase $n_{\mathrm{low}} = \frac{1}{\beta}$; $n_{\mathrm{high}} = 1$. The area below corresponds to uniform low density phase $n_{\mathrm{low}} = n_{\mathrm{high}} < n_{\mathrm{crit}}$. There is no uniform high-density phase like in [5, 8] because the particle-hole symmetry is broken.

One can easily compute the current flowing through the truck. The phase diagram 'current versus $\frac{1}{\beta}$' (narrowness of the road) is given in figure 8. With fixed β, the current j increases with the density n, as

$$j = n(\langle v_{\mathrm{truck}}\rangle + \langle v_{\mathrm{car}}\rangle) = n\left(\frac{1}{\beta}\ \frac{\alpha(1-n)(1-n\beta) + n(\alpha+\beta-n\beta)}{(1-n)(1-n\beta) + n(\alpha+\beta-n\beta)} + (1-n)\right) \tag{27}$$

until the critical density

$$n_{\mathrm{crit}} = \frac{1}{\beta} \tag{28}$$

is reached. After that, in the jammed phase, the current remains constant (see equation (26)),

$$j_{\mathrm{max}} = n_{\mathrm{high}}(1 - n_{\mathrm{high}} + \langle v_{\mathrm{truck}}\rangle) = \frac{1}{\beta} \tag{29}$$

for all densities $n_{\mathrm{crit}} \leqslant n \leqslant 1$.

Note that there is a single critical density value equation (28), in constrast to the models with fixed blockage [5, 8], where two critical densities exist, ρ_{crit} and $\tilde{\rho}_{\mathrm{crit}} = 1 - \rho_{\mathrm{crit}}$. The reason is as follows. The models considered in [5, 8] have the particle-hole symmetry which is broken in the model we consider.

6. The k-point correlation functions

Here we obtain the k-point equal time correlation functions in the steady state, in exact and asymptotic forms, for a system with one truck and M cars. Analogously to equation (16), one defines

$$\langle n(x_1)n(x_2)\ldots n(x_k)\rangle = \frac{\sum_{\mathrm{conf}} \tau_{p_1}\tau_{p_2}\ldots\tau_{p_k}\ w_M(\tau_1\tau_2\ldots\tau_{N-1}2)}{\sum_{\mathrm{conf}} w_M(\tau_1\tau_2\ldots\tau_{N-1}2)} \tag{30}$$

with $p_j = N - 1 - x_j$.

Here we take $x_1 < x_2 < \ldots < x_k$†. Sums run over all possible configurations having M cars, $N - M - 1$ empty spaces, with the truck at the position N.

Calculation of equation (30) leads to the following surprising result:

$$\langle n(x_1)n(x_2)\ldots n(x_k)\rangle = \langle n(x_k)\rangle - \sum_{j=2}^{k} f_j(x_{k+1-j}). \tag{31}$$

The k-point correlation function actually splits into a sum of k terms, each one depending on a single argument! The exact form of $f_j(x)$ is given by

$$f_j(x) = \Bigg[C_{N-2-j}^{M+1-j} + \Bigg\{-\frac{\alpha}{\beta-1}C_{N-j-2}^{M-j} + \frac{\alpha+\beta-1}{\beta-1}[(\beta-1)J(N-j, M-j, x-1) \\ \times\Theta(x \geqslant 2) + \beta C_{N-j-2}^{M-j}]\Bigg\}\Theta(x \geqslant 1)\Bigg]/(Y(N, M)\alpha\beta^M) \tag{32}$$

where $\Theta(x \geqslant y)$ and $Y(N, M)$ are given by equations (19) and (11), respectively, and

$$J(N, M, x) = \sum_{i=1}^{\min(M,x)} \beta^i C_{N-2-i}^{M-i}.$$

We shall show that in the limit of large $N, M \gg 1$, $f_j(x)$ is given by a remarkably simple formula

$$f_j(x) = \kappa^{j-1}(1 - \langle n(x)\rangle) \qquad \text{with } \kappa = \min\left(n, \frac{1}{\beta}\right). \tag{33}$$

Indeed let us consider the connected two-point correlation function,

$$\langle n(x_1)n(x_2)\rangle_C = \langle n(x_1)n(x_2)\rangle - \langle n(x_1)\rangle\langle n(x_2)\rangle = \langle n(x_2)\rangle(1 - \langle n(x_1)\rangle) - f_2(x_1) \tag{34}$$

according to equation (31). As $f_2(x_1)$ does not depend on x_2, one can choose any convenient x_2. Take the point x_2 infinitely far apart from x_1; $x_2 \gg x_1$, so that the correlations between them vanish $\langle n(x_1)n(x_2)\rangle_C = 0$. Then:

(a) before the transition $n\beta \leqslant 1$, we have for $N, M \gg 1$, using equations (20) and (34),

$$f_2(x_1) = n(1 - \langle n(x_1)\rangle).$$

(b) After the transition $n\beta > 1$, using equation (24) and imposing in addition $x_2 \gg Nl$ we obtain $\langle n(x_2)\rangle = \frac{1}{\beta}$, and

$$f_2(x_1) = \frac{1}{\beta}(1 - \langle n(x_1)\rangle)$$

which proves formula (33) for $j = 2$. Recursively, one obtains the asymptotic behaviour of other functions $f_3(x), \ldots, f_k(x)$ from (31). Formula (33) can also be obtained directly from (32).

Finally, the k-point correlation function, connected part, is given from equations (31) and (33) as

$$\langle n(x_1)\ldots n(x_k)\rangle_C = \langle n(x_k)\rangle - \sum_{j=2}^{k}\kappa^{j-1}(1 - \langle n(x_{k+1-j})\rangle) - \langle n(x_1)\rangle\ldots\langle n(x_k)\rangle \\ \text{where } \kappa = \min\left(n, \frac{1}{\beta}\right) \tag{35}$$

† Equality of some argument values, say, $x_1 = x_2$ simply lowers the order of the correlation function by 1 as seen from equation (30); $\langle n(x_1)n(x_1)n(x_3)\ldots n(x_k)\rangle = \langle n(x_1)n(x_3)\ldots n(x_k)\rangle$.

for $N, M \gg 1$, $x_1 < x_2 < \ldots < x_k$ and $\langle n(x)\rangle$ is given by (20) and (24) for $n\beta < 1$ and $n\beta > 1$ respectively. Thus, the k-point correlation function for $N, M \gg 1$ is determined completely by the one-point correlation functions.

As an example, consider the two-point correlation function,

$$\langle n(x_1)n(x_2)\rangle_C = (\langle n(x_2)\rangle - \kappa)(1 - \langle n(x_1)\rangle).$$

Before the transition, at the low-density phase $n\beta < 1$, $\kappa = n$, $\langle n(x_2)\rangle$ is given by (20), and $\langle n(x_2)\rangle - \kappa \sim (n\beta)^{x_2}$. So the correlation function decays exponentially with a length scale

$$\xi = |\ln(n\beta)|^{-1}.$$

Thus, in the low-density phase, the two-point correlation function is nonzero only in a close vicinity of truck $x_1 < x_2 \sim \xi$. For $x_2 \gg \xi$, $\langle n(x_1)n(x_2)\rangle_C \equiv 0$. Thus, in the whole region $\xi \ll x_2 < N$, cars do not feel any correlations between each other and behave like an ideal gas of particles.

In the traffic jam phase, $n\beta > 1$, the two-point correlation function

$$\langle n(x_1)n(x_2)\rangle_C = \left(\langle n(x_2)\rangle - \frac{1}{\beta}\right)(1 - \langle n(x_1)\rangle) \tag{36}$$

remains nonzero, only if both x_1 and $x_2 (x_1 < x_2)$ are in the region $x_1, x_2 \in [Nl - \Delta\sqrt{N}, Nl + \Delta\sqrt{N}]$ as is seen from (24). Otherwise, either $\langle n(x_2)\rangle \approx \frac{1}{\beta}$, or $\langle n(x_1)\rangle \approx 1$ and the correlation function $\langle n(x_1)n(x_2)\rangle_C$ vanishes.

Again, one can say that the jammed phase indeed has a phase separation: (a) solid-like phase with density 1; (b) ideal gas phase (no correlations between the particles-cars) of density $\frac{1}{\beta}$. This explains why the simple hydrodynamic approach (see section 5.3) leads to the correct results in the thermodynamic limit.

7. The bound state between two trucks

Here we shall consider the system having two trucks, and M cars, and determine the probability $\Omega(R)$ of two trucks being at distance R apart. This probability is proportional to

$$\Omega(R) \sim \sum_{\text{conf}} w_M(\tau_1\tau_2 \ldots \tau_{N-R-2} 2 \tau_{N-R} \ldots \tau_{N-1} 2) \tag{37}$$

with the sum running over all possible configurations having M cars and two trucks at positions $(N - R - 1)$ and N. So $R = 0$ corresponds to two trucks being next to each other. Due to the periodic boundary condition, $0 \leqslant R \leqslant \frac{N-1}{2}$.

The exact expression for $\Omega(R)$ at finite N is unwieldy and we shall not present it here (typical behaviours of $\Omega(R)$ are shown in figure 9 for $N = 200, n = 0.3$, and $\beta = 3, 5$). Instead we shall write down its asymptotics in each phase.

(i) $n\beta < 1$:In the thermodynamic limit, $\Omega(R)$ reduces to the following

$$\Omega(R) \sim 1 + \frac{n(1-n)(\alpha+\beta-1)(\alpha-1)}{(1-n+\alpha n)^2}(n\beta)^R. \tag{38}$$

It is maximal for $R = 0$ and decays exponentially with the same length scale $\xi = |\ln(n\beta)|^{-1}$, as before. As ξ does not depend on N, the fraction of space with nonzero correlations between the trucks vanishes as $1/N$. Thus, two trucks are asymptotically free. The asymptotic freedom of trucks accounts for the fact that the phase transition to the jammed phase takes place at the same critical density, $n_{\text{crit}}\beta = 1$. These arguments can be extended to any finite number of trucks in the infinite system $N \to \infty$.

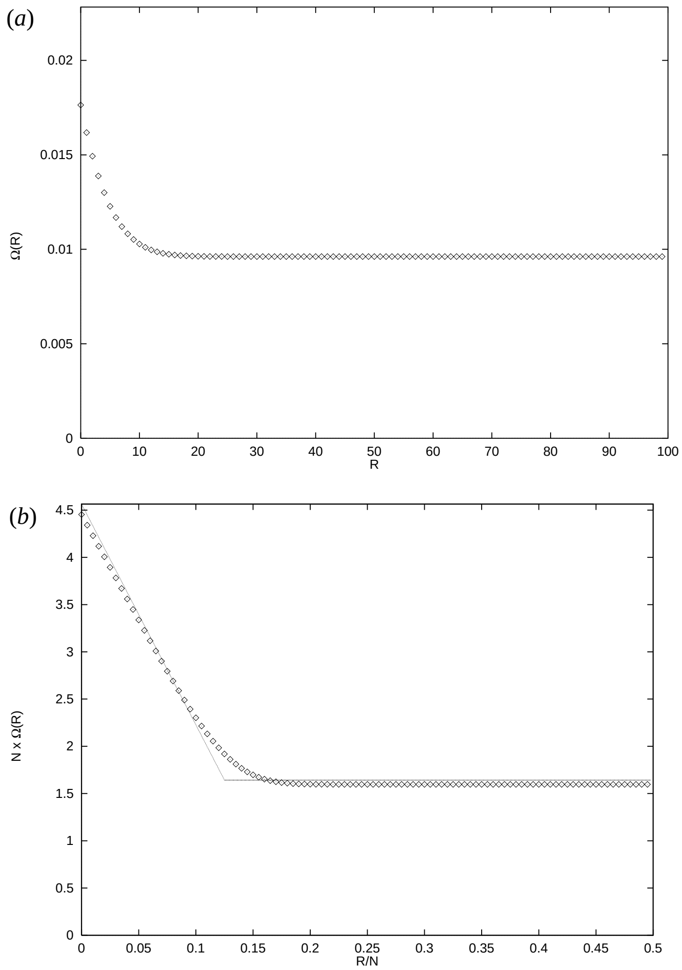

Figure 9. Probability $\Omega(R)$ of finding two trucks at a distance R apart, (a) before and (b) after the phase transition, computed from the exact formula, for $N = 200$, $n = 0.3$. The light curve in (b) corresponds to the thermodynamic limit. The value of β is 3 and 5 for (a) and (b), respectively.

(ii) Jammed phase $n\beta \geqslant 1$: In the thermodynamic limit, we find the following result for $\Omega(R)$ from the exact formula:

$\Omega(R)$ linearly drops with the distance R in the region

$$0 \leqslant R \leqslant Nr_0 \qquad r_0 = \min(l, 1-l) < \tfrac{1}{2} \qquad l = \frac{n\beta - 1}{\beta - 1}$$

and then stays constant $\Omega(R) \equiv$ constant, $Nr_0 \leqslant R \leqslant \frac{N-1}{2}$, see figure 9($b$). The relative ratios are

$$\text{for } r_0 = l \qquad \frac{\Omega(Nr_0)}{\Omega(0)} = 1 - \frac{(\alpha-1)(\beta-1)}{\alpha\beta} \tag{39}$$

$$\text{for } r_0 = 1 - l \qquad \frac{\Omega(Nr_0)}{\Omega(0)} = 1 - \frac{(\alpha-1)(\beta-1)}{\alpha\beta}\frac{r_0}{1-r_0}. \tag{40}$$

One can interprete this as the two trucks forming a weak bound state in the traffic jam phase. The probability $\Omega(R)$ was also studied in [7], for a system with two second-class/many first-class particles, both hopping in the same direction, where it shows a power law decay $\Omega(R) \sim R^{-3/2}$ in the uniform background of the first-class particles. This is in marked contrast to our result equation (38) showing exponential decay in the uniform low-density phase. Note, however, that the asymptotic equation (38) was obtained in supposition $\alpha \neq 1, \beta \neq 1$, while the results of [7] are derived for the $\alpha = 1, \beta = 1$ case.

It is interesting to analyse the average distance between the two trucks. Consider the case $n < 0.5$ first. Analysis of (38) and (39) shows that in the thermodynamic limit the relative distance between the trucks $\frac{\langle r \rangle}{N} = \frac{1}{4}$ in the low-density phase $n\beta < 1$ and then drops monotonically as a function of β from $\frac{\langle r \rangle}{N} = \frac{1}{4}$ to $\frac{\langle r \rangle}{N} = n/3$ at $\beta = \infty$ (complete blockage). At the same time the length $l = (n\beta - 1)/(\beta - 1)$ increases from $l = 0$ at the transition point $\beta_{\mathrm{crit}} = 1/n$ to $l = n$ at $\beta = \infty$. We remind the reader that $l = \frac{n\beta - 1}{\beta - 1}$ is the total length of traffic jam in a system with one truck in the thermodynamic limit, see (22). One can argue in a different way that the total length of the traffic jam is independent of the number of trucks, as long as it remains *finite* (e.g. by using the hydrodynamic approach, see section 5.3). That means that in the early stage of traffic jam phase (small β) $\frac{\langle r \rangle}{N} > l$ and two separate traffic jams in front of the two trucks are formed, of lengths l_1 and l_2, $l_1 + l_2 = l$ separated by the low-density regions. As β increases, eventually $\frac{\langle r \rangle}{N} < l$, and the two separate jams merge into a single one of length l.

8. Summary

We have formulated the two-lane traffic model and showed that it has the transition from the low-density phase to the segregated (traffic jam) phase. Modifying the model to an exactly solvable one, we studied in detail the characteristics of the solvable model, which we believe to be qualitatively correct for the original one. The solvable model in the traffic jam phase in the $N \gg 1$ limit is a special case of the two-species model considered by Derrida [9], see the discussion after equation (23). However, our angle of view is different and most results obtained in sections 4–7 are new. We have obtained exact expressions of the current j, the average density profile, and the k-point correlation functions, for the finite chain, and in the large N limit, for a single truck. We have also studied the two-trucks case and observed that a weakly bound state is formed between them in the traffic jam phase. Generally, the truck slowing down the car movement, can be thought of as a sort of moving blockage. Qualitatively our results for the density profile, the current j phase diagram, two-point correlation functions agree with those obtained in [8] and in part with those in [5], describing a fixed blockage. However, the last two systems possess the particle-hole symmetry and therefore the uniform high-density phase, related to the low-density one by this symmetry. In our case, the particle-hole symmetry is broken for both our original two-lane model and the modified solvable one. This accounts for the absence of the uniform high-density phase in our model, see the phase diagram in figure 8.

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Appendix. Computation of $Y(N, M)$ and $Y_D(N, M)$

The probability of finding the truck at the site N is given up to normalization by

$$Y(N, M) = \sum_{\text{conf}} w_M(\tau_1\tau_2 \ldots \tau_{N-1}2) \tag{A1}$$

where this sum is over all possible configurations having M cars, $N - M - 1$ empty spaces, with the truck at the position N. Split the above sum into two terms as

$$Y(N, M) = \sum_{\text{conf}} w_M^D(\tau_1\tau_2 \ldots \tau_{N-2}12) + \sum_{\text{conf}} w_M^A(\tau_1\tau_2 \ldots \tau_{N-2}02). \tag{A2}$$

The first (second) term corresponds to a car (empty space) being at site $(N-1)$. The second term can be written as

$$w_M^A(\tau_1\tau_2 \ldots \tau_{N-2}02) = \mathrm{Tr}(CAE) = \frac{1}{\alpha}\,\mathrm{Tr}(CA) \tag{A3}$$

using the algebra (7). Here, C corresponds to an arbitrary configuration of length $N - 2$ having M cars and $Q = N - M - 2$ empty spaces. Generally,

$$CA = D^{m_1}A^{q_1}D^{m_2}A^{q_2} \ldots D^{m_k}A^{q_k}$$

with $m_1 + m_2 + \ldots + m_k = M$, $q_1 + q_2 + \ldots + q_k = Q$, $q_k \geqslant 1$.

According to (7),

$$\mathrm{Tr}(CA) = \mathrm{Tr}\left(\frac{1}{\beta^{m_1+m_2+\cdots+m_k}} A^{q_1+q_2+\ldots+q_k}\right) = \frac{1}{\beta^M}\,\mathrm{Tr}(A^Q) \tag{A4}$$

and

$$w_M^A(\tau_1\tau_2 \ldots \tau_{N-2}02) = \frac{1}{\alpha\beta^M}\,\mathrm{Tr}(A^Q). \tag{A5}$$

The first term in (A2) for some specific configuration is

$$w_M^D(\tau_1\tau_2 \ldots \tau_{N-2}12) = \mathrm{Tr}(D^{m_1}A^{q_1}D^{m_2}A^{q_2} \ldots D^{m_{k-1}}A^{q_{k-1}}D^{m_k}E) \tag{A6}$$

where $m_1 + m_2 + \ldots + m_k = M$, $q_1 + q_2 + \ldots + q_{k-1} = Q$, $m_k \geqslant 1$. We have the following recursive relation:

$$f_m = AD^mE = AD^{m-1}(DE) = AD^{m-1}(D + E) = AD^m + f_{m-1}. \tag{A7}$$

Using the last expression recursively, one finds

$$f_m = A\sum_{i=1}^{m} D^i + f_0 \qquad f_0 = AE = \frac{1}{\alpha}A. \tag{A8}$$

Substituting the value of f_m into (A6), using (7), we obtain

$$w_M^D = \frac{1}{\alpha\beta^M}\gamma_{m_k}\,\mathrm{Tr}(A^Q) \tag{A9}$$

where

$$\gamma_m = \alpha\sum_{i=1}^{m}\beta^{m-i} + \beta^m = -\frac{\alpha}{\beta - 1} + \frac{\alpha + \beta - 1}{\beta - 1}\beta^m. \tag{A10}$$

The same common factor $\mathrm{Tr}(A^Q)$ cancels from all formulae for averages starting from (8), (9) etc, as it enters to both the denominator and numerator. Below we shall set $\mathrm{Tr}(A^Q) = 1$ for simplicity.

Using (A5) and (A9) and some combinatorics to count the number of configurations, the sum (A2) then reads

$$Y(N, M) = \frac{1}{\alpha\beta^M} \sum_{m=1}^{M} \gamma_m C_{N-m-2}^{M-m} + \frac{1}{\alpha\beta^M} C_{N-2}^{M}. \tag{A11}$$

Finally, substituting (A10) and performing the summation $\sum_{m=1}^{M} C_{N-m-2}^{M-m} = C_{N-2}^{M-1}$, one obtains equations (11) and (10).

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